Stochastic multiscale modeling of subsurface and surface flows. Part I: Multiscale mortar mixed finite elements for Darcy flow

Ivan Yotov
Department of Mathematics, University of Pittsburgh

KAUST WEP Workshop
January 30–February 1, 2010

Joint work with Todd Arbogast, Gergina Pencheva, Sunil Thomas, and Mary F. Wheeler, The University of Texas at Austin; Benjamin Ganis, University of Pittsburgh
Energy and environment

- Ground water and surface water contamination
- Hydrocarbon energy production
Reservoir rock
Mathematical and numerical challenges for modeling porous media

- Multiscale - space and time
- Multiphysics - aquifer, surface water, waterflood, CO$_2$, polymer, geomechanics
- Multiphase - gas, aqueous, multiple flowing phases
- Highly nonlinear coupled PDE systems - advection, reaction, diffusion/dispersion, capillary effects
- Complex geology and geometry - faults, fractures, layers
- Very large scale computations
  - millions of unknowns
  - parallel computing
Multiblock approach for multiphysics problems

- complex geometry and gridding
- multinumerics

- multiscale resolution
- parallel algorithms

Water flood
Well
Fault
CO₂ flood

Time splitting scheme
Fully implicit model

Department of Mathematics, University of Pittsburgh
Groundwater flow in a faulted aquifer

Numerical grids and low conductivity barriers
Contaminated groundwater flow
DNAPL concentration at 44 days
Outline

• Background and motivation

• A multiscale mortar mixed finite element method

• A priori error estimates

• A domain decomposition algorithm

• A posteriori error estimates

• Mortar and subdomain adaptivity

• A relationship between multiscale mortar MFE methods and subgrid upscaling methods

• A multiscale flux basis formulation

• Extension to two-phase flow
Single phase flow model

\[ u = -K \nabla p \quad \text{in } \Omega \subset \mathbb{R}^d \ (d = 2, 3) \quad \text{(Darcy’s law)} \]

\[ \nabla \cdot u = f \quad \text{in } \Omega \quad \text{(conservation of mass)} \]

\[ u \cdot n = 0 \quad \text{on } \partial \Omega \quad \text{(no flow BC)} \]

Variational mixed formulation

\[ H(\text{div}; \Omega) = \{ v : v \in (L^2(\Omega))^d, \nabla \cdot v \in L^2(\Omega) \} \]

\[ V = \{ v \in H(\text{div}; \Omega) : v \cdot n = 0 \text{ on } \partial \Omega \} \]

\[ W = L^2_0(\Omega) = \{ w \in L^2(\Omega) : \int_{\Omega} w \, dx = 0 \}. \]

Find \( u \in V, \ p \in W \) such that

\[ (K^{-1}u, v) = (p, \nabla \cdot v), \quad v \in V, \]

\[ (\nabla \cdot u, w) = (f, w), \quad w \in W. \]
The mixed finite element method

$\mathcal{T}_h$ - finite element partition

$V_h \times W_h \subset V \times W$ - mixed finite element spaces

Find $u_h \in V_h, \ p_h \in W_h$ such that

$$(K^{-1}u_h, v) = (p_h, \nabla \cdot v), \quad v \in V_h,$$

$$(\nabla \cdot u_h, w) = (f, w), \quad w \in W_h.$$

- Simultaneous (accurate) approximation of pressure and velocity

- Local mass conservation: for each element $E$,

$$w = \begin{cases} 1 & \text{on } E, \\ 0 & \text{otherwise} \end{cases} \implies \int_E \nabla \cdot u_h = \int_E q.$$

- Continuity of normal flux across element faces: for each $e = \partial E_1 \cap \partial E_2$,

$$u_h|_{E_1} \cdot n_e = u_h|_{E_2} \cdot n_e.$$
Error estimates and convergence testing

**Theorem** [Ingram-Wheeler-Xue-Y.]:  
\[ \| u - u_h \| + \| \nabla \cdot (u - u_h) \| + \| p - p_h \| \leq C h \]
Motivation for multiscale modeling: flow in heterogeneous porous media

Heterogeneous permeability varies on a fine scale. Full fine scale grid resolution ⇒ large, highly coupled system of equations ⇒ solution is computationally intractable

- Variational Multiscale Method
  - Hughes et al; Brezzi
  - Mixed FEM: Arbogast et al

- Multiscale Finite Elements
  - Hou, Wu, Cai, Efendiev et al
  - Mixed FEM: Chen and Hou; Aarnes et al

New approach: based on domain decomposition and mortar finite elements

More flexible - easy to improve global accuracy by refining the local mortar grid where needed
Multiscale finite element/subgrid upscaling methods

\[ L_\epsilon u = f \quad \Rightarrow \quad u \in V : \ a(u, v) = (f, v) \ \forall \ v \in V. \]

Multiscale approximation: \( H \) - coarse grid, \( h \approx \epsilon \) - fine grid (subgrid)

\[ V_{H,h} = V_H + V'_h \]

Basis for \( V'_h(E) \): \( \phi_{h,i}^E, \ i = 1, \ldots, N_E, \)

\[ a_E(\phi_{H,i}^E + \phi_{h,i}^E, v_h) = 0 \quad \forall \ v_h \in V_h(E) \]

Multiscale solution: \( u_{H,h} \in V_{H,h}, \)

\[ a(u_{H,h}, v_{H,h}) = (f, v_{H,h}) \quad \forall \ v_{H,h} \in V_{H,h} \]
Multiblock formulation for single phase flow

\[ \bar{\Omega} = \bigcup_{i=1}^{n} \bar{\Omega}_i; \quad \Gamma_{ij} = \partial \Omega_i \cap \partial \Omega_j \]

On each block \( \Omega_i \):

\[
\begin{align*}
    \mathbf{u} &= -K \nabla p \quad \text{in} \ \Omega_i \\
    \nabla \cdot \mathbf{u} &= q \quad \text{in} \ \Omega_i \\
    \mathbf{u} \cdot \mathbf{n} &= 0 \quad \text{on} \ \partial \Omega_i \cap \partial \Omega
\end{align*}
\]

On each interface \( \Gamma_{ij} \):

\[
\begin{align*}
    p_i &= p_j \quad \text{on} \ \Gamma_{ij} \\
    [\mathbf{u} \cdot \mathbf{n}]_{ij} &= 0 \quad \text{on} \ \Gamma_{ij}
\end{align*}
\]

where

\[
\begin{align*}
    p_i &= p|_{\partial \Omega_i} \\
    [\mathbf{u} \cdot \mathbf{n}]_{ij} &\equiv \mathbf{u}|_{\Omega_i} \cdot \mathbf{n} - \mathbf{u}|_{\Omega_j} \cdot \mathbf{n}
\end{align*}
\]
Multiblock discretization spaces

\[ V_h = \bigoplus_{i=1}^{n} V_{h,i}, \quad W_h = \bigoplus_{i=1}^{n} W_{h,i}, \quad M_h = \bigoplus_{0 \leq i < j \leq n} M_{h,ij} \]

\[ \lambda_h|_{\Gamma_{ij}} \in M_{h,ij}, \quad \int_{\Gamma_{ij}} [u_h \cdot n]_{ij} \mu = 0, \mu \in M_{h,ij}. \]

Subdomain grids do not need to match.
Find $u_h \in V_h$, $p_h \in W_h$, $\lambda_h \in M_h$ s.t. for $1 \leq i \leq n$

\[
(K^{-1}u_h, v)_{\Omega_i} - (p_h, \nabla \cdot v)_{\Omega_i} + \langle \lambda_h, v \cdot n_i \rangle_{\Gamma_i} = 0, \quad v \in V_{h,i},
\]

\[
(\nabla \cdot u_h, w)_{\Omega_i} = (q, w)_{\Omega_i}, \quad w \in W_{h,i},
\]

\[
\sum_{i=1}^{n} \langle u_h \cdot n_i, \mu \rangle_{\Gamma_i} = 0, \quad \mu \in M_h.
\]

Stability, optimal convergence, superconvergence:
Two-scale formulation: mortar upscaling

Two-scale problem:

- Each block is an element of the coarse grid.
- Each block is discretized on the fine scale.
- A coarse mortar space on each interface.
- Result: Effective solution, fine scale on subdomains, coarse scale flux matching
Multiscale mortar mixed finite element method

Allow for different scales and polynomial approximations on interfaces and subdomains.

Assume

\[ P_k \subset V_{h,i}, \quad P_l \subset W_{h,i}, \quad P_m \subset M_H, \quad m \geq k + 1 \]

Find \( u_h \in V_h, \ p_h \in W_h, \ \lambda_H \in M_H \) s.t. for \( 1 \leq i \leq n \)

\[
(K^{-1} u_h, v)_{\Omega_i} - (p_h, \nabla \cdot v)_{\Omega_i} + \langle \lambda_H, v \cdot n_i \rangle_{\Gamma_i} = 0, \quad v \in V_{h,i},
\]

\[
(\nabla \cdot u_h, w)_{\Omega_i} = (q, w)_{\Omega_i}, \quad w \in W_{h,i},
\]

\[
\sum_{i=1}^{n} \langle u_h \cdot n_i, \mu \rangle_{\Gamma_i} = 0, \quad \mu \in M_H.
\]

Stability assumption:

\[
\| \mu \|_{0, \Gamma_{i,j}} \leq C(\| Q_{h,i} \mu \|_{0, \Gamma_{i,j}} + \| Q_{h,j} \mu \|_{0, \Gamma_{i,j}}), \quad \mu \in M_H, \quad 1 \leq i < j \leq n.
\]
An approximation result

Weakly continuous velocities:

\[ \mathbf{V}_{h,0} = \left\{ \mathbf{v} \in \mathbf{V}_h : \sum_{i=1}^{n} \langle \mathbf{v} |_{\Omega_i} \cdot \mathbf{n}_i, \mu \rangle_{\Gamma_i} = 0 \quad \forall \quad \mu \in M_H \right\}. \]

Equivalent formulation: find \( \mathbf{u}_h \in \mathbf{V}_{h,0} \) and \( p_h \in W_h \) such that

\[
(K^{-1} \mathbf{u}_h, \mathbf{v}) - (p_h, \nabla \cdot \mathbf{v}) = 0, \quad \mathbf{v} \in \mathbf{V}_{h,0},
\]

\[
(\nabla \cdot \mathbf{u}_h, w) = (q, w), \quad w \in W_h
\]

Interpolation operator \( \Pi_0 : \mathbf{V} \to \mathbf{V}_{h,0} \) such that

\[
(\nabla \cdot (\Pi_0 q - q), w)_\Omega = 0, \quad w \in W_h.
\]

\[
\|\Pi_0 q - q\|_0 \leq C \sum_{i=1}^{n} (\|q\|_{r,\Omega_i} h^r + \|q\|_{r+1/2,\Omega_i} h^r H^{1/2}), \quad 1 \leq r \leq k + 1
\]
A priori error estimates

Theorem:

\[ \| u - u_h \| \leq C (H^{m+1/2} + h^{k+1}), \quad \| \nabla \cdot (u - u_h) \| \leq Ch^{l+1} \]

\[ \| | u - u_h | | \| \leq C (H^{m+1/2} + h^{k+1} H^{1/2}) \]

\[ \| p - p_h \| \leq C (H^{m+3/2} + h^{k+1} H + h^{l+1}) \]

\[ \| | p - p_h | | \| \leq C H \| u - u_h \|_{H(div)} \]

Balance \( \| u - u_h \| \) or \( \| | p - p_h | | \| \) error terms (with \( l = k \)):

\[ H = h^{\frac{k+1}{m+1/2}} \Rightarrow \| u - u_h \| \leq Ch^{k+1}, \quad \| | p - p_h | | \| \leq C h^{k+1 + \frac{k+1}{m+1/2}} \]

For example, RT_0, \( k = 0 \), and quadratic mortars, \( m = 2 \),

\[ H = h^{2/5} : \| u - u_h \| \leq Ch, \quad \| p - p_h \| \leq Ch, \quad \| | p - p_h | | \| \leq C h^{1+2/5} \]
Parallel domain decomposition

Two types of subdomain problems:

\[ g_H(\mu) = \sum_{i=1}^{n} \langle \bar{u}_{h,i} \cdot n_i, \mu \rangle_{\Gamma_i} \]

\[ a_{H,i}(\lambda, \mu) = -\langle u_{h,i}^{*}(\lambda) \cdot n_i, \mu \rangle_{\Gamma_i} \]

\[ a_H(\lambda, \mu) = \sum_{i=1}^{n} a_{H,i}(\lambda, \mu) \]

The solution \((u_h, p_h, \lambda_H)\) to the original problem satisfies

\[ A_H \lambda_H = g_H \quad \text{or} \quad a_H(\lambda_H, \mu) = g(\mu), \quad \forall \mu \in M_H, \]

with \(u_h = u_h^{*}(\lambda_H) + \bar{u}_h, \) \(p_h = p_h^{*}(\lambda_H) + \bar{p}_h.\)
Interface iteration

**Lemma** The interface operator $A_H : M_H \rightarrow M_H$ is symmetric and positive semi-definite.

$$a_{H,i}(\lambda, \mu) = (K^{-1}u_{h,i}^*(\lambda), u_{h,i}^*(\mu)).$$

$A_H : \lambda_H \rightarrow [u_h^*(\lambda_H) \cdot n]$ is a Steklov-Poincare operator.

Apply the Conjugate Gradient method for $A_H \lambda_H = g_H$.

Computing the action of the operator (needed at each CG iteration):

- Given mortar data $\lambda_H \in M_H$, project onto subdomain grids:
  $$\lambda_H \rightarrow Q_{h,i} \lambda_H$$

- Solve local problems in parallel with boundary data $Q_{h,i} \lambda_H$

- Project fluxes onto the mortar space and compute the jump:
  $$u_{h,i} \cdot n_i \rightarrow Q_{h,i}^T u_{h,i} \cdot n_i, \quad A_H \lambda_H = Q_{h_1}^T u_{h,1} \cdot n_1 + Q_{h_2}^T u_{h,2} \cdot n_2$$
Numerical experiments

| $m$ | $H$   | $||p - p_h||$ | $||u - u_h||$ | $||p - p_h||$ | $||u - u_h||$ | $||p - \lambda_H||$ |
|-----|-------|---------------|---------------|---------------|---------------|---------------------|
| 2   | $h^{1/2}$ | 1             | 1             | 1.5           | 1.25          | 1.25                |
| 1   | $2h$   | 1             | 1             | 2             | 1.5           | 1.5                 |

<table>
<thead>
<tr>
<th></th>
<th>full $K$</th>
<th>diag $K$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2h^{1/2}$</td>
<td>1.25</td>
<td>1.5</td>
</tr>
<tr>
<td>$2h$</td>
<td>1.5</td>
<td>2</td>
</tr>
</tbody>
</table>

Table 1: Theoretical convergence rates for quadratic and linear mortars.

Example 1:

$$p(x, y) = x^3 y^4 + x^2 + \sin(xy)\cos(y), \quad K = \begin{pmatrix} (x + 1)^2 + y^2 & \sin(xy) \\ \sin(xy) & (x + 1)^2 \end{pmatrix}.$$  

Example 2:

$$p(x, y) = \begin{cases} x^2 y^3 + \cos(xy) & \text{if } 0 \leq x \leq 1/2, \\ \left(\frac{2x+9}{20}\right)^2 y^3 + \cos\left(\frac{2x+9}{20}y\right) & \text{if } 1/2 \leq x \leq 1 \end{cases}, \quad K = \begin{cases} I, & \text{if } 0 \leq x \leq 1/2, \\ 10 \times I, & \text{if } 1/2 \leq x \leq 1 \end{cases}.$$
Computed solution for Example 1

A. Discontinuous quadratic mortars

B. Discontinuous linear mortars

Computed pressure (shade) and velocity (arrows).
## Convergence rates for Example 1

| $1/h$ | iter. | $| p - p_h |$ | $| u - u_h |$ | $| p - p_h |$ | $| u - u_h |$ | $| p - \lambda_H |$ |
|-------|-------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 4     | 8     | 2.64E-1         | 2.03E-1         | 4.62E-2         | 2.13E-2         | 4.45E-2         |
| 16    | 13    | 6.37E-2         | 4.86E-2         | 2.83E-3         | 1.81E-3         | 2.72E-3         |
| 64    | 15    | 1.59E-2         | 1.21E-2         | 1.75E-4         | 1.60E-4         | 1.69E-4         |
| 256   | 16    | 3.98E-3         | 3.03E-3         | 1.09E-5         | 1.77E-5         | 1.08E-5         |

**rate**

- $O(h^{1.01})$  
- $O(h^{1.01})$  
- $O(h^{2.01})$  
- $O(h^{1.71})$  
- $O(h^{2.00})$

**Continuous quadratic mortars on non-matching grids**

| $1/h$ | iter. | $| p - p_h |$ | $| u - u_h |$ | $| p - p_h |$ | $| u - u_h |$ | $| p - \lambda_H |$ |
|-------|-------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 4     | 4     | 2.63E-1         | 2.04E-1         | 4.54E-2         | 2.35E-2         | 4.55E-2         |
| 8     | 7     | 1.28E-1         | 9.82E-2         | 1.14E-2         | 7.32E-3         | 1.13E-2         |
| 16    | 13    | 6.37E-2         | 4.86E-2         | 2.82E-3         | 2.23E-3         | 2.83E-3         |
| 32    | 18    | 3.18E-2         | 2.43E-2         | 7.01E-4         | 6.95E-4         | 7.05E-4         |
| 64    | 23    | 1.59E-2         | 1.21E-2         | 1.75E-4         | 2.24E-4         | 1.76E-4         |
| 128   | 23    | 7.95E-3         | 6.06E-3         | 4.36E-5         | 7.47E-5         | 4.40E-5         |
| 256   | 24    | 3.98E-3         | 3.03E-3         | 1.09E-5         | 2.54E-5         | 1.09E-5         |

**rate**

- $O(h^{1.01})$  
- $O(h^{1.01})$  
- $O(h^{2.00})$  
- $O(h^{1.65})$  
- $O(h^{2.00})$

**Continuous linear mortars on non-matching grids**
Error in the computed solution for Example 1

A. Discontinuous quadratic mortars

B. Discontinuous linear mortars

Error in computed pressure (shade) and velocity (arrows).
Iterative convergence for Example 2

<table>
<thead>
<tr>
<th>$1/h$</th>
<th><strong>BalCG</strong></th>
<th></th>
<th></th>
<th><strong>CG</strong></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\text{cond.}$</td>
<td>$\text{iter.}$</td>
<td>$\text{cond.}$</td>
<td>$\text{iter.}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>$1.83E+0$</td>
<td>5</td>
<td>$1.45E+1$</td>
<td>8</td>
<td></td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>$2.49E+1$</td>
<td>13</td>
<td>$4.29E+1$</td>
<td>20</td>
<td></td>
<td></td>
</tr>
<tr>
<td>64</td>
<td>$2.33E+1$</td>
<td>14</td>
<td>$1.27E+2$</td>
<td>29</td>
<td></td>
<td></td>
</tr>
<tr>
<td>256</td>
<td>$2.96E+1$</td>
<td>15</td>
<td>$3.63E+2$</td>
<td>45</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Continuous quadratic mortars on matching grids

<table>
<thead>
<tr>
<th>$1/h$</th>
<th><strong>BalCG</strong></th>
<th></th>
<th></th>
<th><strong>CG</strong></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\text{cond.}$</td>
<td>$\text{iter.}$</td>
<td>$\text{cond.}$</td>
<td>$\text{iter.}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>$1.79E+1$</td>
<td>5</td>
<td>$3.91E+1$</td>
<td>8</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>$1.78E+1$</td>
<td>8</td>
<td>$3.74E+1$</td>
<td>11</td>
<td></td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>$2.50E+1$</td>
<td>13</td>
<td>$3.82E+1$</td>
<td>19</td>
<td></td>
<td></td>
</tr>
<tr>
<td>32</td>
<td>$3.68E+1$</td>
<td>19</td>
<td>$6.60E+1$</td>
<td>26</td>
<td></td>
<td></td>
</tr>
<tr>
<td>64</td>
<td>$4.71E+1$</td>
<td>23</td>
<td>$1.30E+2$</td>
<td>34</td>
<td></td>
<td></td>
</tr>
<tr>
<td>128</td>
<td>$5.96E+1$</td>
<td>24</td>
<td>$2.58E+2$</td>
<td>51</td>
<td></td>
<td></td>
</tr>
<tr>
<td>256</td>
<td>$7.29E+1$</td>
<td>24</td>
<td>$5.16E+2$</td>
<td>72</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Continuous linear mortars on matching grids
A posteriori error estimates

- Estimate the error by computable quantities
- Use the error estimator to dynamically adapt the grids

\[ E \in T_h : \quad \omega^2_E = \|K^{-1}u_h + \nabla p_h\|^2_E h_E^2 + \|f - \nabla \cdot u_h\|^2_E h_E^2 + \|\lambda_H - p_h\|^2_{\partial E \cap \Gamma} h_E, \]

\[ \tau \in T^{\Gamma,H} : \quad \omega^2_\tau = \|[u_h \cdot n]\|^2_{\tau} H^3_\tau, \]

\[ \tilde{\omega}^2_E = h_E^{-2} \omega^2_E, \quad \tilde{\omega}^2_\tau = H^{-2}_\tau \omega^2_\tau. \]

Theorem (Upper bounds):

\[ \|p - p_h\|^2 \leq C \left\{ \sum_{E \in T_h} \omega^2_E + \sum_{\tau \in T^{\Gamma,H}} \omega^2_\tau \right\}, \]

\[ \|u - u_h\|^2_{H(\text{div})} \leq C \left\{ \sum_{E \in T_h} \tilde{\omega}^2_E + \sum_{\tau \in T^{\Gamma,H}} \tilde{\omega}^2_\tau \right\}. \]
Residual-based estimates: lower bounds

Theorem:

\[
\sum_{E \in T_h} \omega_E^2 + \sum_{\tau \in T_{\Gamma,H}} \omega_\tau^2 \leq C \left( \| p - p_h \|^2 + \sum_{E \in T_h} h_E^2 \| u - u_h \|^2_{H(\text{div};E)} \right)
\]

\[
+ \sum_{\tau \in T_{\Gamma,H}} H_\tau \| \lambda - \lambda_H \|^2_\tau + \sum_{\tau \in T_{\Gamma,H}} h_{E,\tau}^{-1} H_\tau^3 \| u - u_h \|^2_{H(\text{div};E_\tau)}
\]

\[
\sum_{E \in T_h} \tilde{\omega}_E^2 + \sum_{\tau \in T_{\Gamma,H}} \tilde{\omega}_\tau^2 \leq C \left( \sum_{E \in T_h} h_E^{-2} \| p - p_h \|^2_E + \| u - u_h \|^2_{H(\text{div})} \right).
\]

Efficient (lower) and reliable (upper) estimate for \( p - p_h \):

\[
C_1 \left( \sum_{E \in T_h} \omega_E^2 + \sum_{\tau \in T_{\Gamma,h}} \omega_\tau^2 \right) \leq \| p - p_h \|^2 \leq C_2 \left( \sum_{E \in T_h} \omega_E^2 + \sum_{\tau \in T_{\Gamma,H}} \omega_\tau^2 \right).
\]
Adaptive mesh refinement algorithm

1. Solve the problem on a coarse (both subdomain and mortar) grid.

2. For each subdomain $\Omega_i$
   
   (a) Compute
   
   $$\omega_i = \left( \sum_{E \in T_{h,i}} \omega_E^2 + \sum_{\tau \in T_{\Gamma_{i,h}}} \omega_\tau^2 \right)^{1/2}$$
   
   (b) If $\omega_i > 0.5 \max_{1 \leq j \leq n} \omega_j$, refine $T_{h,i}$.

3. For each interface $\Gamma_{i,j}$, if either $\Omega_i$ or $\Omega_j$ has been refined $m$ times, refine $T_{h,i,j}$.

4. Solve the problem on the refined grid. If either the desired error tolerance or the maximum refinement level has been reached, exit; otherwise go to step 2.
Numerical experiments

Example 3: 2D problem with a boundary layer

\[ p(x, y) = 1000 \, x \, y \, e^{-10(x^2+y^2)}, \quad K = I \]

Dirichlet BCs; Continuous quadratic mortars

Example 4: 2D problem with highly oscillating tensor

\[
K = \begin{cases} 
(105 - 100 \sin(20\pi x) \sin(20\pi y)) \ast I, & 0 \leq x, y \leq 1/2 \text{ or } 1/2 \leq x, y \leq 1 \\
(105 - 100 \sin(2\pi x) \sin(2\pi y)) \ast I, & \text{otherwise}
\end{cases}
\]

BCs: \( p|_{x=0} = 1, \quad p|_{x=1} = 0 \), no flow on the rest of the boundary

Discontinuous quadratic mortars

Multiblock decomposition: 6 × 6 subdomains

Coarse grid: 2 × 2 on each subdomain and adaptive mesh refinement
Computed solution for Example 3

Pressure on the fourth grid level

Discontinuous quadratic mortars

Discontinuous linear mortars
Computed solution for Example 4

Magnitude of the velocity on the fifth grid level

Discontinuous quadratic mortars

Discontinuous linear mortars
Relation to subgrid upscaling methods

Subgrid upscaling (Arbogast et. al.):

\[ V_h = V_{h,1} + V_{h,2} + V_H \]

Multiscale mortar MFE:

\[ V_{h,0} = V_{h,1} + V_{h,2} : \langle [v \cdot n], \mu \rangle = 0, \mu \in M_H \]

The two methods are related to the two (dual) non-overlapping domain decomposition formulations of Glowinski and Wheeler.
**Multiscale mortar method: domain decomposition**

The solution \((u_h, p_h, \lambda_H)\) to the original problem satisfies the interface problem

\[ A_H \lambda_H = g_H \quad \text{or} \quad a_H(\lambda_H, \mu) = g(\mu), \quad \forall \mu \in M_H, \]

with \(u_h = u_h^*(\lambda_H) + \bar{u}_h, \quad p_h = p_h^*(\lambda_H) + \bar{p}_h.\)

**CG on the interface:** on each iteration compute \(A_H \lambda_H = -[u_h^*(\lambda_H) \cdot n]\) by solving subdomain problems in parallel.
Note on the implementation

It is possible to solve

\[ A_H \lambda_H = g_H \]

by solving for the discrete Green’s functions

\[ u_h^*(\mu^j_H) \]

for each mortar basis function \( \mu^j_H \in M_H \)

and forming \( A_H \) explicitly.

In this case cost is comparable to subgrid upscaling and multiscale FEM.

The iterative approach is more efficient as long as the number of inerface iterations is less than the number of mortar degrees of freedom per subdomain.
Multiscale flux basis

\[ \left\{ \phi_{H,j}^{(k)} \right\}_{k=1}^{N_{H,j}} : \text{basis for the mortar space } M_{H,j} \text{ on } \Gamma_j. \]

\[ \lambda_{H,j} = \sum_{k=1}^{N_{H,j}} \lambda_{H,j}^{(k)} \phi_{H,j}^{(k)} \]

Multiscale flux basis:

\[ \psi_{H,j}^{(k)} = A_{H,j} \phi_{H,j}^{(k)}, \quad k = 1, \ldots, N_{H,j} \]

Computing \( \psi_{H,j}^{(k)} \) requires solving a subdomain problem in \( D_j \) with Dirichlet boundary data \( \phi_{H,j}^{(k)} \).

Using the pre-computed multiscale flux basis in the CG iteration:

\[ A_{H,j} \lambda_{H,j} = A_{H,j} \left( \sum_{k=1}^{N_{H,j}} \lambda_{H,j}^{(k)} \phi_{H,j}^{(k)} \right) = \sum_{k=1}^{N_{H,j}} \lambda_{H,j}^{(k)} A_{H,j} \phi_{H,j}^{(k)} = \sum_{k=1}^{N_{H,j}} \lambda_{H,j}^{(k)} \psi_{H,j}^{(k)}. \]
Numerical experiments with multiscale flux basis

• Two types of grid refinement are performed
  – **Type A**: Each subdomain has $10 \times 10$ grid; Number of Subdomains is increased
    ![Grid Refinement Type A]
  – **Type B**: Global fine grid is fixed at $120 \times 120$ elements; Number of Subdomains is increased
    ![Grid Refinement Type B]

• Three numerical methods are compared
  – **Method D1**: Original Multiscale Mortar Method Implementation, No preconditioner
  – **Method D2**: Original Multiscale Mortar Method Implementation, Balancing Preconditioner
  – **Method D3**: New Multiscale Flux Basis Implementation, No preconditioner
Table 2: Type A - Continuous Linear Mortars (3x1 grids)

<table>
<thead>
<tr>
<th>Subdomains</th>
<th>Method D1</th>
<th></th>
<th>Method D2</th>
<th></th>
<th>Method D3</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>CGIter</td>
<td>Solves</td>
<td>CGIter</td>
<td>Solves</td>
<td>CGIter</td>
<td>Solves</td>
</tr>
<tr>
<td>2x2x1 = 4</td>
<td>11</td>
<td>14</td>
<td>11</td>
<td>41</td>
<td>11</td>
<td>19</td>
</tr>
<tr>
<td>3x3x1 = 9</td>
<td>19</td>
<td>22</td>
<td>18</td>
<td>64</td>
<td>19</td>
<td>35</td>
</tr>
<tr>
<td>4x4x1 = 16</td>
<td>26</td>
<td>29</td>
<td>22</td>
<td>76</td>
<td>26</td>
<td>35</td>
</tr>
<tr>
<td>5x5x1 = 25</td>
<td>31</td>
<td>34</td>
<td>24</td>
<td>82</td>
<td>31</td>
<td>35</td>
</tr>
<tr>
<td>6x6x1 = 36</td>
<td>38</td>
<td>41</td>
<td>24</td>
<td>82</td>
<td>38</td>
<td>35</td>
</tr>
<tr>
<td>7x7x1 = 49</td>
<td>43</td>
<td>46</td>
<td>25</td>
<td>85</td>
<td>43</td>
<td>35</td>
</tr>
<tr>
<td>8x8x1 = 64</td>
<td>50</td>
<td>53</td>
<td>24</td>
<td>82</td>
<td>50</td>
<td>35</td>
</tr>
</tbody>
</table>

Table 3: Type B - Continuous Linear Mortars (3x1 grids)

<table>
<thead>
<tr>
<th>Subdomains</th>
<th>Method D1</th>
<th></th>
<th>Method D2</th>
<th></th>
<th>Method D3</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>CGIter</td>
<td>Solves</td>
<td>CGIter</td>
<td>Solves</td>
<td>CGIter</td>
<td>Solves</td>
</tr>
<tr>
<td>2x2x1 = 4</td>
<td>11</td>
<td>14</td>
<td>12</td>
<td>44</td>
<td>11</td>
<td>19</td>
</tr>
<tr>
<td>3x3x1 = 9</td>
<td>21</td>
<td>24</td>
<td>19</td>
<td>67</td>
<td>21</td>
<td>35</td>
</tr>
<tr>
<td>4x4x1 = 16</td>
<td>29</td>
<td>32</td>
<td>24</td>
<td>82</td>
<td>29</td>
<td>35</td>
</tr>
<tr>
<td>5x5x1 = 25</td>
<td>34</td>
<td>37</td>
<td>26</td>
<td>88</td>
<td>34</td>
<td>35</td>
</tr>
<tr>
<td>6x6x1 = 36</td>
<td>40</td>
<td>43</td>
<td>26</td>
<td>88</td>
<td>40</td>
<td>35</td>
</tr>
<tr>
<td>7x7x1 = 49</td>
<td>47</td>
<td>50</td>
<td>26</td>
<td>88</td>
<td>47</td>
<td>35</td>
</tr>
<tr>
<td>8x8x1 = 64</td>
<td>53</td>
<td>56</td>
<td>26</td>
<td>88</td>
<td>53</td>
<td>35</td>
</tr>
</tbody>
</table>
Extension to two-phase flow

On each subdomain $\Omega_i$:

$$U_\alpha = -\frac{k_\alpha (S_\alpha) K}{\mu_\alpha} \rho_\alpha (\nabla P_\alpha - \rho_\alpha g \nabla D) \quad \text{(Darcy's law)}$$

$$\frac{\partial (\phi \rho_\alpha S_\alpha)}{\partial t} + \nabla \cdot U_\alpha = q_\alpha \quad \text{(conservation of mass)}$$

On each interface $\Gamma_{ij}$:

$$P_\alpha|_{\Omega_i} = P_\alpha|_{\Omega_j}, \quad [U_\alpha \cdot n]_{ij} = 0.$$ 

On each $\Omega_i$ and $\Gamma_{ij}$:

$$S_w + S_n = 1, \quad p_c(S_w) = P_n - P_w.$$
Domain decomposition

Interface operator $B_H : M_H \rightarrow M_H$

For $\lambda = (P_n^M, P_w^M) \in M_H$

$$B_H(\lambda) = ([U_n^M(\lambda)], [U_w^M(\lambda)],$$

where $U^M_\alpha(\lambda)$ is the mortar projection of the solution $U_\alpha(\lambda) \cdot n$ to subdomain problems with Dirichlet boundary data $\lambda$.

The original problem is equivalent to solving for $\lambda \in M_H$ such that

$$B_H(\lambda) = 0.$$
Computational experiment

SPE Comparative Solution Upscaling Project; Oil-water displacement in a horizontal cross-section $1200 \times 2200$ [ft]

Permeability: $2 \times 10^{-3} - 2 \times 10^5$

Computational grid $60 \times 220$
25 blocks: $5 \times 5$
Computed oil pressure profiles at 2951 days

Three runs: fine grid (1 block), multiblock run with a single linear mortar element on each interface (upscaled), multiblock run with refined mortars (6 elements) near the wells (adapted mortar).
Computed oil concentration profiles at 2951 days

Fine grid solution  Upscaled solution  Adapted mortar solution

Adapted mortar increases production well rates accuracy by a factor of 2 at the cost of increasing CPU time by 50%.
Comparison of linear and quadratic mortars for two-phase flow

A. Permeability field

B. Grids on the coarsest level

A. Oil pressure

B. Water saturation

Comparison of recovery curves.
Summary

- Mortar methods are related to multiscale methods.
- The multiscale mortar MFE method can be viewed as a generalization of subgrid upscaling methods.
- Different scales and polynomial degree approximations on interfaces and subdomains.
- Effective solution: fine subdomain resolution with coarse-grid flux matching.
- Optimal fine scale convergence.
- Multiscale flux basis implementation can increase efficiency.