Adaptive Discontinuous Galerkin Methods for Coupled Diffusion- and Advection-Dominated Transport Phenomena

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Abstract: Adaptive discontinuous Galerkin methods are formulated to solve reactive transport problems, in particular to simulate the coupled diffusion- and advection-dominated phenomena. Numerical results demonstrate that the flexibility of discontinuous Galerkin methods on treating nonconforming meshes can lead to an efficient and effective implementation of dynamic mesh adaptation. In addition, the dynamic approaches resolve time-dependent transport adequately without slope limiting for both long-term and short-term simulations. Moreover, mass conservation is retained locally during dynamic mesh modification. Comparison studies indicate that the anisotropic mesh adaptation provides more efficient meshes and less numerical diffusion than isotropic approaches for problems involving thin layer phenomena.

Keywords: discontinuous Galerkin methods, a posteriori error estimators, anisotropic adaptation, dynamic adaptation, reactive transport.

1. INTRODUCTION

Discontinuous Galerkin (DG) methods are the finite element methods that employ discontinuous piecewise polynomial functions to approximate the solutions of differential equations [24, 10, 5, 3]. Inter-element continuity and boundary conditions are imposed weakly in DG methods, which results in many attractive properties [11, 13, 25, 15, 4]. First of all, the flexibility of DG allows for general non-conforming meshes with variable degrees of approximation. This makes the implementation of h-p adaptivity for DG substantially easier than that for conventional approaches. Moreover, DG methods are locally mass conservative at the element level. In addition, they have less numerical diffusion than most conventional algorithms. They can treat rough coefficient problems and can effectively capture discontinuities in solutions. DG can naturally handle inhomogeneous boundary conditions and curved boundaries. The average of the trace of the fluxes from a DG solution along an element edge is continuous and can be extended so that a continuous flux is defined over the entire domain. Thus DG can be easily coupled with conforming methods. Furthermore, with appropriate meshing, DG with varying p can yield exponential convergence rates. For time-dependent problems in particular, the mass matrices of DG are block diagonal, but not for conforming methods. This provides a substantial computational advantage, especially if explicit time integrations are used.

Transport is a fundamental process arising in many important engineering and scientific fields. Due to involvement of boundary or interior layers, coupled diffusion- and advection-dominated problems have presented significant computational challenges. DG has recently been applied for flow and transport problems in porous media [17, 14, 26, 21, 23], and an optimal convergence in $L^2(\Omega)$ was demonstrated for flow and transport problems [14, 15, 16]. The $hp$-convergence behaviors in the $L^2(L^2)$ and negative norms have also been analyzed [16]. In addition, explicit a posteriori error estimates of DG for reactive transport have been studied [22, 20, 19].

In this paper, we formulate and study three error dynamic mesh adaptation strategies; namely, the hierarchic anisotropic (HA), the $L^2(L^2)$ isotropic (LI), and the hierarchic isotropic (HI). In Section 2, we formulate the general scalar reactive transport equation and corresponding primal DG schemes. In Section 3, we present three a posteriori error estimators, and establish dynamic mesh adaptation strategies. Section 4 is devoted to numerical studies of DG to a coupled diffusion- and advection-dominated problem. Finally, in Section 5, our results are summarized and future work is described.

2. DISCONTINUOUS GALERKIN SCHEMES

2.1. Governing equations. We consider reactive transport for a single flowing phase in porous media. We assume that the Darcy velocity field $\mathbf{u}$ is given and time independent, and it satisfies $\nabla \cdot \mathbf{u} = q$, where $q$ is the imposed external total flow rate.

For simplicity, only a single advection-diffusion-reaction equation is considered. Results for systems with kinetic reactions can be derived by similar arguments. In addition, for convenience, we assume $\Omega$ is a bounded polygonal domain in $\mathbb{R}^d$, $(d = 1, 2$ or $3)$ with boundary $\partial \Omega = \overline{\Gamma}_{\text{in}} \cup \overline{\Gamma}_{\text{out}}$. Here we denote by $\Gamma_{\text{in}}$ the inflow boundary and $\Gamma_{\text{out}}$ the outflow/no-flow boundary, i.e.

\[ \Gamma_{\text{in}} = \{ x \in \partial \Omega : \mathbf{u} \cdot \mathbf{n} < 0 \}, \]
\[ \Gamma_{\text{out}} = \{ x \in \partial \Omega : \mathbf{u} \cdot \mathbf{n} \geq 0 \}, \]

where $\mathbf{n}$ denotes the unit outward normal vector to $\partial \Omega$.

Let $T$ be the final simulation time. The classical transport equation in a single flowing phase in porous media
is given by, for \((x, t) \in \Omega \times (0, T)\),
\[
\frac{\partial c}{\partial t} + \nabla \cdot (uc - D(u)c) = q^* + r(c),
\]
where the unknown variable \(c\) is the concentration of a species (amount per volume). Here \(\phi\) denotes porosity; \(D(u)\) is a dispersion/diffusion tensor; \(r(c)\) is a reaction term; \(q^*\) is a source term, where the imposed external total flow rate \(q\) is a sum of sources (injection) and sinks (extraction); \(c^*\) is an injected concentration \(c_w\) if \(q \geq 0\) and is a resident concentration \(c\) if \(q < 0\).

The following boundary conditions are imposed for this problem:
\[
\begin{align*}
(2.2) \quad (uc - D(u)c) \cdot n &= c_B u \cdot n \quad \text{on } \Gamma_{\text{in}}, \\
(2.3) \quad (-D(u)c) \cdot n &= 0 \quad \text{on } \Gamma_{\text{out}},
\end{align*}
\]
where \(c_B\) is the inflow concentration. The initial concentration is specified in the following way:
\[
(2.4) \quad c(x, 0) = c_0(x) \quad x \in \Omega.
\]

2.2. Notation for discretization. Let \(E_h\) be a family of non-degenerate and possibly non-conforming partitions of \(\Omega\) composed of triangles or quadrilaterals if \(d = 2\), or tetrahedra, prisms or hexahedra if \(d = 3\). Here \(h\) is the maximum element diameter for the mesh. The non-degeneracy requirement (also called regularity) is that the element is convex, and that there exists \(\rho > 0\) such that if \(h_j\) is the diameter of \(E_j \in E_h\), then each of the sub-triangles (for \(d = 2\)) or sub-tetrahedra (for \(d = 3\)) of element \(E_j\) contains a ball of radius \(\rho h_j\) in its interior. We assume that no element crosses the boundaries of \(\Gamma_{\text{in}}\) or \(\Gamma_{\text{out}}\). The set of all interior edges (for 2 dimensional domain) or faces (for 3 dimensional domain) for \(E_h\) is denoted by \(\Gamma_h\). On each edge or face \(\Gamma_h\), a unit normal vector \(n_{\gamma}\) is chosen. The sets of all edges or faces on \(\Gamma_{\text{in}}\) and on \(\Gamma_{\text{in}}\) for \(E_h\) are denoted by \(\Gamma_{\text{h,in}}\) and \(\Gamma_{\text{h,out}}\), respectively, for which the normal vector \(n_{\gamma}\) coincides with the outward unit normal vector.

For \(s \geq 0\), we define:
\[
(2.5) \quad H^s(\Omega) = \{ \phi \in L^2(\Omega) : \phi|_E \in H^s(E), E \in E_h \}.
\]
The discontinuous finite element space is taken to be
\[
(2.6) \quad D_r(\Omega) = \{ \phi \in L^2(\Omega) : \phi|_E \in \mathbb{P}_r(E), E \in E_h \},
\]
where \(\mathbb{P}_r(E)\) denotes the space of polynomials of (total) degree less than or equal to \(r\) on \(E\). Note that the results in this paper are based on the local space \(\mathbb{P}_r\), but can be easily extended to the local space \(Q_r\).

We now define the average and jump for \(\phi \in H^s(\Omega)\), \(s > 1/2\). Let \(E_i, E_j \in E_h\) and \(\gamma = \partial E_i \cap \partial E_j \in \Gamma_h\) with \(n_{\gamma}\) exterior to \(E_i\). Denote
\[
(2.7) \quad \{ \phi \} := \frac{1}{2} \left( \phi|_{E_i} \right)_\gamma + \left( \phi|_{E_j} \right)_\gamma,
\]
\[
(2.8) \quad [\phi] := \left( \phi|_{E_i} \right)_\gamma - \left( \phi|_{E_j} \right)_\gamma.
\]
Denote the upwind value of concentration \(c^*\) as follows:
\[
c^*_\gamma := \begin{cases} 
  c_{E_i} & \text{if } u \cdot n_{\gamma} \geq 0 \\
  c_{E_j} & \text{if } u \cdot n_{\gamma} < 0.
\end{cases}
\]
The usual Sobolev norm on \(\Omega\) is denoted by \(\|\cdot\|_{m, \Omega}\) [1]. The broken norms are defined, for a positive integer \(m\), as
\[
(2.9) \quad \|\phi\|^2_{m, E} := \sum_{E \in \mathcal{E}_h} \|\phi\|^2_{m, E}.
\]
The inner product in \((L^2(\Omega))^d\) or \(L^2(\Omega)\) is indicated by \((\cdot, \cdot)_\gamma\) and the inner product in the boundary function space \(L^2(\gamma)\) is denoted by \((\cdot, \cdot)_\gamma\). The “cut-off” operator \(M\) is defined as
\[
(2.10) \quad M(c)(x) := \min(c(x), M),
\]
where \(M\) is a large positive constant.

2.3. Continuous-in-time schemes. We introduce the bilinear form \(B(c, w; u)\) defined as
\[
(2.11) \quad B(c, w; u) := \sum_{E \in \mathcal{E}_h} \int_E (D(u)c - cu) \cdot \nabla w - \int_\Omega c q^- w \\
+ J^e_0(c, w) - \sum_{\gamma \in \Gamma_h} \left( \int_{\gamma} (D(u)c \cdot n_{\gamma}) \right) [w]
\]
\[
- s_{\text{form}} \sum_{\gamma \in \Gamma_h} \int_{\gamma} (D(u)c) \nabla w \cdot n_{\gamma} \right) [c]
\]
\[
+ \sum_{\gamma \in \Gamma_{h, \text{out}}} \int_{\gamma} c^* u \cdot n_{\gamma} [w] + \int_{\gamma} c u \cdot n_{\gamma} w,
\]
where \(s_{\text{form}} = -1\) for NIPG (the Nonsymmetric Interior Penalty Galerkin method [15]) or OBB-DG (the Oden-Babuška-Baumann formulation of DG [11]), \(s_{\text{form}} = 1\) for SIPG (the Symmetric Interior Penalty Galerkin method [18, 24, 16]) and \(s_{\text{form}} = 0\) for IIPG (the Incomplete Interior Penalty Galerkin method [18, 16, 8]). Here \(q^+\) is the injection source term and \(q^-\) is the extraction source term, i.e.,
\[
q^+ = \max(q, 0), \quad q^- = \min(q, 0).
\]
By definition, we have \(q = q^+ + q^-\). In addition, we define the interior penalty term \(J^e_0(c, w)\) as
\[
(2.12) \quad L(w; u, c) := \int_\Omega r (M(c)) w + \int_\Omega c w q^+ w + \sum_{\gamma \in \Gamma_{h, \text{in}}} \int_{\gamma} c_B u \cdot n_{\gamma} w,
\]
where \(\sigma\) is a discrete positive function that takes the constant value \(\sigma_{\gamma}\) on the edge or face \(\gamma\). We let \(\sigma_{\gamma} \equiv 0\) for OBBD-DG, and assume \(0 < \sigma_0 \leq \sigma_{\gamma} \leq \sigma_m\) for SIPG, NIPG and IIPG.

The linear functional \(L(w; u, c)\) is defined as
\[
(2.13) \quad \left( \frac{\partial \phi C^D_{DG}}{\partial t}, w \right) + B(C^D_{DG}, w; u)
\]
\[
= L(w; u, C^D_{DG}), \quad w \in \mathcal{V}_r(\mathcal{E}_h), \quad t \in (0, T],
\]
\[
(2.14) \quad c C^D_{DG}, w = (\phi_0, w), w \in \mathcal{V}_r(\mathcal{E}_h), \quad t = 0.
\]
3. Dynamic Mesh Adaptation

3.1. Residual-based a posteriori error estimators.
We introduce residual quantities that only depend on the approximate solution and the data. The residuals consist of the interior residual $R_{I}$, the zeroth order boundary residual $R_{B0}$ and the first order boundary residual $R_{B1}$ defined below:

$$
R_{I} = q^{C_{DG}*} + r(M(C_{DG})) - \frac{\partial C_{DG}}{\partial t} - \nabla \cdot (C_{DG} u - D(u) \nabla C_{DG}),
$$

(3.1)

$$
R_{B0} = \begin{cases} 
[C_{DG}], & \text{on } \Gamma_{h}, \\
0, & \text{on } \partial \Omega,
\end{cases}
$$

(3.2)

$$
R_{B1} = \begin{cases} 
[D(u) \nabla C_{DG} \cdot n], & \text{on } \Gamma_{h}, \\
(c_{B} u - C_{DG} u + D(u) \nabla C_{DG}) \cdot n, & \text{on } \Gamma_{h, in}, \\
D(u) \nabla C_{DG} \cdot n, & \text{on } \Gamma_{h, out},
\end{cases}
$$

(3.3)

where, in element interior, $C_{DG}*$ = $c$ if $q < 0$ and $C_{DG}*$ = $c_{w}$ if $q \geq 0$.

We recall two a posteriori error estimators [22, 19].

Theorem 3.1. (Explicit a posteriori error estimator in $L^2(L^2)$ for SIPG) Under certain regularity assumptions [19], there exists a constant $K$, independent of $h$ and $r$, such that

$$
\|C_{DG} - c\|_{L^2(0,T;L^2)} \leq K \left( \sum_{E \in \mathcal{E}_{h}} \eta_{E}^2 \right)^{1/2},
$$

where

$$
\eta_{E}^2 = \frac{h_{E}}{r} \|R_{I}\|_{L^2(0,T;L^2(E))}^2 
+ \frac{1}{2} \sum_{\gamma \in \partial E \setminus \partial \Omega} \left( h_{\gamma} \|R_{B0}\|_{L^2(0,T;L^2(\gamma))} \right) 
+ \frac{1}{2} \sum_{\gamma \in \partial E \setminus \partial \Omega} h_{\gamma} \|R_{B1}\|_{L^2(0,T;L^2(\gamma))}^2 
+ \frac{1}{2} \sum_{\gamma \in \partial E \setminus \partial \Omega} \frac{h_{\gamma}^2}{r^{1/2}} \|R_{B1}\|_{L^2(0,T;L^2(\gamma))}.
$$

Here $h_{\gamma} = \max(h_{E_{1}}, h_{E_{2}})$ for $\gamma = \partial E_{1} \cap \partial E_{2}$, $r \geq 1$ and $\delta = 0$ in the case of conforming meshes with triangles or tetrahedra. In general, $\delta = 1$.

Theorem 3.2. (Explicit a posteriori error estimator for OBB-DG, NIPG, SIPG or IIPG) Under certain regularity assumptions [22], there exists a constant $K$, independent of $h$, such that

$$
\|\sqrt{\delta}(C_{DG} - c)\|_{L^\infty(0,T;L^2(\Omega))} 
+ \|D^{\frac{1}{2}}(u) \nabla (C_{DG} - c)\|_{L^2(0,T;L^2(\Omega))} 
\leq K \left( \sum_{E \in \mathcal{E}_{h}} \eta_{E}^2 \right)^{1/2},
$$

where

$$
\eta_{E}^2 = \frac{h_{E}^2}{r} \|R_{I}\|_{L^2(0,T;L^2(E))}^2 
+ \sum_{\gamma \in \partial E \setminus \partial \Omega} h_{\gamma} \|R_{B1}\|_{L^2(0,T;L^2(\gamma))}^2 
+ \frac{1}{2} \sum_{\gamma \in \partial E \setminus \partial \Omega} \left( h_{\gamma} \|R_{B1}\|_{L^2(0,T;L^2(\gamma))} \right) 
+ h_{\gamma} \|R_{B0}\|_{L^2(0,T;L^2(\gamma))}^2 
+ h_{\gamma} \|R_{B1}\|_{L^2(0,T;L^2(\gamma))}.
$$

Here $h_{\gamma} = \max(h_{E_{1}}, h_{E_{2}})$ for $\gamma = \partial E_{1} \cap \partial E_{2}$ and $r \geq 1$.

We note that, in implementation, time derivative terms may be approximated by finite differences. For example, the time derivative of the DG solution used in the interior residual $R_{I}$ may be approximated by

$$
\frac{\partial C_{DG}}{\partial t} \approx \frac{(C_{DG})_{t_{n+1}} - (C_{DG})_{t_{n}}}{t_{n+1} - t_{n}} 	ext{ for } t \in (t_{n}, t_{n+1}).
$$

3.2. An a posteriori error indicator using hierarchic bases. Residual-based explicit error estimators are efficient to compute, and can be used to indicate the subset of elements that need to be refined, thus achieving adaptivity. However, these residual-based estimators yield only one piece of information for each element, and thus do not provide the guidance on anisotropic refinements. In this section, we derive error estimators using hierarchic bases. Unlike residual-based error estimators, hierarchic error estimators give point-wise information of the error, and thus can be used to guide fully anisotropic $hp$-adaptation. Here, for simplicity, we consider only the anisotropic $h$-adaptation.

Hierarchic error estimators consist of solving the problem of interest employing two discretization schemes of different accuracy and using the difference in the approximations as an estimate for the error. The advantages of this approach include their applicability to many classes of problems and the simplicity and ease of their implementation. The reader is referred to [2, 6, 7, 9, 12] for further information.

For a given mesh $\mathcal{E}_{h}$, we construct the mesh $\mathcal{E}_{h/2}$ by isotropically refining each element of $\mathcal{E}_{h}$. We denote by $C_{DG}$ the DG solution in the coarse mesh $\mathcal{E}_{h}$, and by $C_{DG,F}$ the DG solution in the fine mesh $\mathcal{E}_{h/2}$. We make the following saturation assumption:

$$
\|C_{DG,F} - c\|_{L^2} \leq \beta \|C_{DG} - c\|_{L^2} \text{, } 0 \leq \beta < 1.
$$

(3.4)

Using the a priori convergence results of DG [18, 16, 14], we observe, for all the four primal DGs, i.e. OBB-DG, SIPG, NIPG and IIPG, that $\beta$ is less than or equal to the following asymptotic value:

$$
\beta \leq \frac{1}{2^{r-1}} = \frac{1}{2^{\min(r,s)-1}}.
$$

Theorem 3.3. (A posteriori error estimator for OBB-DG, SIPG, NIPG or IIPG) Let the saturation
where the mesh is adaptively adjusted, without changing the growing dynamic adaptive strategy presented in [20], Algorithm 3.4. (dynamic mesh adaptation)

3.3. Dynamic mesh adaptation. Here we extend the non-growing dynamic adaptive strategy presented in [20], where the mesh is adaptively adjusted, without changing the number of elements. The initial mesh is chosen to be a uniform fine grid. We denote by \#(S) the number of elements in a set S.

Algorithm 3.4. (dynamic mesh adaptation)

Given an initial mesh \(\mathcal{E}_0\), a modification factor \(\alpha \in (0, 1)\), time slices \(\{(T_0, T_1), (T_1, T_2), \ldots, (T_{N-1}, T_N)\}\), and iteration numbers \(\{M_1, M_2, \ldots, M_N\}\).

1. Let \(n = 1\);
2. Let \(m = 1\);
3. Compute the initial concentration for time slice \((T_{n-1}, T_n)\) using either the initial condition (if \(n = 1\)) or the concentration at the end of last time slice (if \(n > 1\)) by the \(L^2\) local projection;
4. Let \(\mathcal{E}_{m,n} = \mathcal{E}_0\) if \(n = 1\) and \(m = 1\); or \(\mathcal{E}_{m,n} = \mathcal{E}_{M_{n-1}+1,n-1}\) if \(n > 1\) and \(m = 1\);
5. Compute the DG approximation of the PDE for time slices \((T_{n-1}, T_n)\) based on mesh \(\mathcal{E}_{m,n}\) and compute the error indicator \(\eta_E\) for each element \(E \in \mathcal{E}_{m,n}\);
6. Select \(\mathcal{E}_r \in \mathcal{E}_{m,n}\) such that \(#(\mathcal{E}_r) = \lceil \alpha \#(\mathcal{E}_{m,n}) \rceil\) and \(\min\{\eta_E : E \in \mathcal{E}_r\} \geq \max\{\eta_E : E \in \mathcal{E}_{m,n} \setminus \mathcal{E}_r\}\);
7. Select \(\mathcal{E}_c \in \mathcal{E}_{m,n}\) to minimize \(\max\{\eta_E : E \in \mathcal{E}_c\}\) subject to \(#(\mathcal{E}_c) = \lceil \alpha \#(\mathcal{E}_{m,n}) \rceil\) and that \(\mathcal{E}_c\) satisfies the coarsening-compatible condition with regard to \(\mathcal{E}_{m,n}\) and \(\mathcal{E}_r\);
8. Refine all \(E \in \mathcal{E}_r\) isotropically (or anisotropically guided by the local error indicator), and coarsen all \(E \in \mathcal{E}_c\) to form a new mesh \(\mathcal{E}_{m+1,n}\);
9. Let \(m = m + 1\). If \(m \leq M_n\), go to step 3;
10. Let \(n = n + 1\). If \(n \leq N\), go to step 2;
11. Report the solution and stop.

We have used a coarsening-compatible condition in the above algorithm, which is defined as below.

Definition 3.5. (Coarsening-compatible condition)

Coarsening element set \(\mathcal{E}_c\) is said to satisfy the coarsening-compatible condition with regard to the mesh \(\mathcal{E}\) and refinement element set \(\mathcal{E}_r\) if and only if:
1. Each element in \(\mathcal{E}_c\) has a father;
2. Brothers of an element in \(\mathcal{E}_c\) are active, that is, they sit in \(\mathcal{E}_r\);
3. None of the elements in \(\mathcal{E}_c\) and their brothers are in \(\mathcal{E}_r\);
4. Brothers of an element in \(\mathcal{E}_c\) are not in \(\mathcal{E}_r\).

4. Numerical Results

In this section, we investigate three dynamic mesh adaptation strategies using numerical experiments. These adaptation strategies are listed as follows.

- LI approach: the dynamic and isotropic mesh adaptation using the \(L^2(L^2)\) norm error indicator \(\eta_E\) in Theorem 3.1;
- HI approach: the dynamic and isotropic mesh adaptation using the error indicator \(\eta_E\) of hierarchical bases in Theorem 3.3;
- HA approach: the dynamic and anisotropic mesh adaptation using the error indicator \(\eta_E\) of hierarchical bases in Theorem 3.3.

We solve equation (2.1) over the domain \(\Omega = (0, 1)^2\). We impose no injection or extraction, i.e. \(q = 0\). The initial and inflow boundary concentrations are both zero. The porosity \(\phi\) is a constant 0.1, and the diffusion-dispersion tensor \(D\) is a diagonal matrix with \(D_{ii} = 0.01\). The domain is divided into three layers, i.e. the lower one \(\Omega_1 = (0, 1) \times (0, 0.25)\), the middle layer \(\Omega_m = (0.1) \times (0.25, 0.5)\), and the upper half \(\Omega_m = (0, 1) \times (0.5, 1)\). The velocity is \(\mathbf{u}\) is zero in \(\Omega_m\) and is \((1, 0)\) uniformly elsewhere. The reaction term is taken to be a constant source term: \(r(c) = s\), and the contaminant source term \(s\) is 1 in \(\Omega_m\) and is zero.

![Figure 4.1. Velocity \(\mathbf{u}\) and contaminant source rate \(s\) (\(s = 1\) in the red area and \(s = 0\) in the blue areas)]
contain 10 time steps. The implicit Euler method is employed for time integration. The initial mesh is a 16x16 uniform rectangular grid. Concentration profiles and mesh structures from the three approaches are shown in Figures 4.2, 4.3 and 4.4 respectively. For each adaptation approach, we run a long-term simulation up to $t=10$ using a uniform time step $\Delta t=0.1$ (shown in the two subfigures on the top of each figure), and a short-term simulation up to $t=0.01$ using a uniform time step $\Delta t=0.0001$ (shown in the two subfigures on the top of each figure).

Results clearly indicate that concentration boundary layers are formed almost immediately. The boundary layers are very thin at early times and grow spatially with time. The concentration profile changes rapidly in early times, but slows as time increases. It ultimately approaches a steady state, which is reached shortly after $t=2$. The local physics of diffusion and advection is captured in all three adaptation approaches. The top third part of the domain (i.e. $y \geq 2/3$) has nearly zero concentration during early times, and very little diffusion or advection takes place. Correspondingly, this area has very coarse grids for all three approaches. The successful application of DG to this example occurs because of the effectiveness of DG in treating both advection- and diffusion-dominated problems. In addition, DG offers flexible and effective mesh adaptation in capturing boundary layer phenomena.

Both the HI and the HA approaches yield meshes that are dense only around boundary layers necessary for resolving the physics. On the other hand, the LI approach unnecessarily refines the mesh in the center of the middle layer, away from the boundary layers; this is especially true in early times. It should be noted that the boundary layers expand during early simulation times, and both the HI and the HA lead to meshes well adapted to this time-evolving phenomena. The HA approach does not only provide proper density distribution of the mesh, but also yields physics-driven element aspect ratios. During early times, most of the elements are narrow, which mimics the shape of the thin boundary layer. During later times,
the elements are still slim in the advection-dominated regions, but are more or less isotropic in the diffusion region. This example illustrates physics-driven mesh adaptation, in which advection is direction-oriented while diffusion is isotropic. It should be observed that the corner points \((0, 0.25)\) and \((0.5, 0)\), being the starting points of the boundary layers, play important roles in boundary layer formation. We note that both HI and HA resolve the starting regions of the boundary layers adequately.

5. Discussion and Conclusions

We have numerically investigated three dynamic adaptation strategies, namely, HA, LI, and HI. Results indicate that all approaches resolve time-dependent transport processes for both long- and short-term simulations, and eliminate the need of slope limiters. Numerical experiments demonstrate that DG can treat both advection- and diffusion-dominated problems, and anisotropic mesh adaptations are flexible and effective in capturing boundary (interior) layer phenomena. Our numerical results further show that the number of iterations in each time slice can be as small as 1 or 2 in obtaining an accurate mesh. We emphasize that, due to the discontinuous spaces employed in DG, the projections of concentration during mesh modifications only involve local computations and are locally mass conservative. These features ensure both the efficiency and the accuracy of DG during dynamic mesh modifications. Comparisons of the three dynamic adaptation approaches clearly demonstrate the superior effectiveness of HA, which results in the most efficient physics-driven meshes and has the least numerical diffusion. One of our future research directions is to approximate the hierarchical error indicator by solutions of only local problems on a fine grid, hopefully reducing the computational cost while maintaining its superior numerical performance.

References